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Resolution of an apparent inconsistency in the electromagnetic Casimir effect

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Abstract

The vacuum expectation value of the electromagnetic energy–momentum tensor between two parallel plates in spacetime dimensions $D > 4$ is calculated in the axial gauge. While the pressure between the plates agrees with the global Casimir force, the energy density is divergent at the plates and not compatible with the total energy which follows from the force. However, subtracting the divergent self-energies of the plates, the resulting energy is finite and consistent with the force. In analogy with the corresponding scalar case for spacetime dimensions $D > 2$, the divergent self-energy of a single plate can be related to the lack of conformal invariance of the electromagnetic Lagrangian for dimensions $D > 4$.

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Two parallel, metallic plates separated by the distance L in vacuum will interact due to the modifications of the quantum fluctuations of the electromagnetic field caused by the boundary conditions at the plates. The resulting force was first calculated by Casimir [1] who found it to be given by the attractive pressure $P = -\pi^2/240L^4$. Using the conformal symmetry of the electromagnetic field in $D = 4$ spacetime dimensions, Brown and Maclay [2] later obtained the vacuum expectation values of all the components of the electromagnetic energy–momentum tensor

$$T_{\mu\nu} = F_{\mu\alpha}F_{\nu}^{\alpha} - \eta_{\mu\nu}\mathcal{L} \quad (1)$$

where $\mathcal{L} = -(1/4)F_{\alpha\beta}^2$ is the standard Lagrangian. While these expectation values were constant between the plates, the corresponding fluctuations of the separate electric and magnetic fields were found by Lütken and Ravndal to be in general non-constant and actually divergent as one approaches one of the plates [3]. These divergences are caused by imposing ideal boundary conditions valid for arbitrarily small wavelengths of the field. A physical boundary would only affect fluctuations down to a finite wavelength which is expected to result in an increasing, but finite value of the fluctuations near the plates. The quantitative

effects of such more realistic boundary conditions have been investigated during the last few years but a complete and satisfactory description is still lacking [4].

Casimir forces in spacetimes with dimensions $D > 4$ were first systematically calculated by Ambjørn and Wolfram [5]. For the electromagnetic field between two parallel hyperplanes with separation L , the attractive pressure was found to be

$$P = -(D-1)(D-2) \frac{\Gamma(D/2)\zeta_R(D)}{(4\pi)^{D/2}L^D} \quad (2)$$

where the factor $D-2$ is the number of physical degrees of freedom in the field resulting from gauge invariance. If the energy density between the plates is constant, it would just be this pressure divided by the factor $D-1$. This is the case when $D=4$ and it is of interest to see if it holds also in the more general case $D > 4$. For this purpose we calculate in the following the separate fluctuations of the electric and magnetic components of the field which then allows us to find all the vacuum expectation values of the components of the energy-momentum tensor (1).

Today these quantum effects could be of relevance for stacks of parallel branes where the electromagnetic field is replaced by one or more of the Abelian Ramond-Ramond fields. Any divergent energy density would then have serious implications for the stability of such configurations due to the resulting large gravitational interactions.

The electromagnetic field tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ in $D = d+1$ spacetime dimensions has d electric components $E_i = F_{0i}$ and $d(d-1)/2$ magnetic components $B_{ij} = F_{ij}$. For the geometry under consideration, the simplest and most natural choice of gauge is the axial gauge $n^\mu A_\mu = 0$ where the unit D -vector n^μ is normal to the plates. Taking this along the z -axis, we thus have $A_z = 0$. The component A_0 is no longer a free variable, but depends on the others via the Maxwell equation $\partial_i F^{i0} = 0$. It gives $A_0 = -\Delta^{-1} \partial_i \dot{A}_i$ where the operator $\Delta = \partial_i^2$. There are thus $D-2$ independent degrees of freedom described by the spatial field components A_i where $i \neq z$. The full Lagrangian then follows as

$$L = \frac{1}{2} \int d^d x [\dot{A}_i (\delta_{ij} - \partial_i \Delta^{-1} \partial_j) \dot{A}_j - A_i (\partial_i \partial_j - \delta_{ij} \Delta) A_j] \quad (3)$$

after a few partial integrations and neglecting surface terms.

In order to quantize the system, we must solve the classical wave equation following from the Lagrangian. For this purpose we impose the boundary condition $n^\mu F_{\mu\nu} = 0$ at the plates. This is the same as for the MIT quark bag where it had a physical justification [6]. Here it is just taken for convenience. In the axial gauge it gives $\partial_z A_i = 0$ at the plates which is the Neumann boundary condition for each physical field component $A_i(x) = A_i(t; \mathbf{x}_T, z)$. We then have the general mode expansion

$$A_i(t; \mathbf{x}_T, z) = \sqrt{\frac{2}{L}} \sum_{n=1}^{\infty} \int \frac{d^{d-1} k_T}{(2\pi)^{d-1}} A_{in}(t, \mathbf{k}_T) e^{i\mathbf{k}_T \cdot \mathbf{x}_T} \cos\left(\frac{n\pi z}{L}\right) \quad (4)$$

which satisfies the wave equation and the boundary conditions. The factor $\sqrt{2/L}$ is a normalization factor. In the mode sum we have dropped a term with $n=0$ since it will not contribute to any physical results after regularization.

Quantization can now be done in the standard way. We introduce orthonormal polarization vectors \mathbf{e}_λ normal to the wavevector \mathbf{k}_T and a longitudinal polarization vector \mathbf{e}_L along this direction. The coordinate components A_{in} of the field are then replaced by the polarization components $(A_{\lambda n}, A_{Ln})$. After quantization at $t=0$ the transverse components can then be written in the standard form as

$$A_{\lambda n}(\mathbf{k}_T) = \sqrt{\frac{1}{2\omega_n}} [a_{\lambda n}(\mathbf{k}_T) + a_{\lambda n}^\dagger(-\mathbf{k}_T)] \quad (5)$$

where $\omega_n^2 = \mathbf{k}_T^2 + k_z^2$ with $k_z = \pi n/L$. The creation and annihilation operators now have the standard commutator

$$[a_{\lambda n}(\mathbf{k}_T), a_{\lambda' n'}(\mathbf{k}'_T)] = \delta_{\lambda\lambda'} \delta_{nn'} (2\pi)^{d-1} \delta(\mathbf{k}_T - \mathbf{k}'_T). \quad (6)$$

However, the longitudinal component

$$A_{Ln}(\mathbf{k}_T) = \sqrt{\frac{1}{2\omega_n} \left(\frac{\omega_n}{k_z}\right)} [a_{Ln}(\mathbf{k}_T) + a_{Ln}^\dagger(-\mathbf{k}_T)] \quad (7)$$

contains an extra factor when the corresponding creation and annihilation operators have the same canonical commutator (6). The full field operator (4) is then expressed in terms of these new operators corresponding to definite polarization states.

The field fluctuations between the two plates can now easily be calculated. As a simple example, consider $E_z = -\partial_z \Delta^{-1} \partial_j \dot{A}_j$. If we isolate the mode with quantum numbers (n, \mathbf{k}_T) , we find the operator

$$\Delta^{-1} \partial_j \dot{A}_j = \sqrt{\frac{2}{L}} \sqrt{\frac{1}{2\omega_n} \frac{(ik_j)(-i\omega_n)}{\omega_n^2}} \left[a_{\lambda n} e_{\lambda j} + \left(\frac{\omega_n}{k_z}\right) a_{Ln} e_{Lj} \right] e^{i\mathbf{k}_T \cdot \mathbf{x}_T} \cos\left(\frac{n\pi z}{L}\right) + \text{H.c.} \quad (8)$$

acting on the vacuum state. Here we have used that Δ gives $\mathbf{k}_T^2 + k_z^2 = \omega_n^2$ in momentum space. We see that the transverse modes will not contribute here since they satisfy the orthogonality condition $\mathbf{e}_\lambda \cdot \mathbf{k}_T = 0$. However, for the longitudinal mode we have instead $\mathbf{e}_L \cdot \mathbf{k}_T = k_T$ and it will give a non-zero contribution. The derivative ∂_z gives a factor k_z and $\cos(n\pi z/L) \rightarrow \sin(n\pi z/L)$. For this mode alone we thus get the fluctuation

$$\langle E_z^2 \rangle_{n, \mathbf{k}_T} = \frac{2}{L} \frac{1}{2\omega_n} k_T^2 \sin^2 \frac{n\pi z}{L} \quad (9)$$

Including all the modes, we thus have for the full fluctuation of this electric field component

$$\langle E_z^2 \rangle = \frac{1}{L} \sum_{n=1}^{\infty} \int \frac{d^{d-1} k_T}{(2\pi)^{d-1}} \left[\omega_n - \frac{k_z^2}{\omega_n} \right] \sin^2 \frac{n\pi z}{L} \quad (10)$$

when we write $k_T^2 = \omega_n^2 - k_z^2$. For the other components we similarly find

$$\langle E_i^2 \rangle = \frac{1}{L} \sum_{n=1}^{\infty} \int \frac{d^{d-1} k_T}{(2\pi)^{d-1}} \left[\omega_n (d-2) + \frac{k_z^2}{\omega_n} \right] \cos^2 \frac{n\pi z}{L} \quad (11)$$

where there is an implied sum over the transverse index i . The magnetic field fluctuations can be obtained the same way and become

$$\langle B_{iz}^2 \rangle = \frac{1}{L} \sum_{n=1}^{\infty} \int \frac{d^{d-1} k_T}{(2\pi)^{d-1}} \left[\omega_n + \frac{k_z^2}{\omega_n} (d-2) \right] \sin^2 \frac{n\pi z}{L} \quad (12)$$

$$\langle B_{i<j}^2 \rangle = \frac{1}{L} \sum_{n=1}^{\infty} \int \frac{d^{d-1} k_T}{(2\pi)^{d-1}} \left[\omega_n (d-2) - \frac{k_z^2}{\omega_n} (d-2) \right] \cos^2 \frac{n\pi z}{L} \quad (13)$$

when we again sum over the indices i and j . We have also confirmed these results by performing the same calculations in Coulomb gauge instead of axial gauge.

Using now a combination of dimensional and zeta-function regularization as previously used when $D = 4$ [7], we can write the result in the form

$$\begin{aligned} & \frac{1}{2L} \sum_{n=1}^{\infty} \int \frac{d^{d-1} k_T}{(2\pi)^{d-1}} \left(1 \pm \cos \frac{2n\pi z}{L} \right) \left\{ \frac{\omega_n}{k_z^2 / \omega_n} \right\} \\ & = -\frac{\Gamma(D/2)}{(4\pi)^{D/2} L^D} \left[\zeta_R(D) \pm \frac{1}{2} f_D(z/L) \right] \left\{ \frac{1}{D-1} \right\}. \end{aligned} \quad (14)$$

Here $\zeta_R(D)$ is the Riemann zeta-function while $f_D(z/L)$ depends on the distance z from the plates. When the spacetime dimension D is even, it can be written in the compact form

$$f_D(z/L) = \frac{\pi^D}{\Gamma(D)} \left(-\frac{d}{d\theta} \right)^{D-1} \cot \theta \quad (D = \text{even}) \quad (15)$$

where $\theta = \pi z/L$. But when D is odd, no such closed expression is easily derived. However, using a different regularization based on the corresponding point-split Green functions, one finds in general [8]

$$f_D(z/L) = \sum_{j=-\infty}^{\infty} \frac{1}{|j + z/L|^D} = \zeta_H(D, z/L) + \zeta_H(D, 1 - z/L) \quad (16)$$

where $\zeta_H(D, z/L)$ is the Hurwitz zeta-function. When D is even, this can be shown to agree with (15).

The regularized fluctuations of the electric field normal to the plates thus become

$$\langle E_z^2 \rangle = \frac{(D-2)\Gamma(D/2)}{(4\pi)^{D/2}L^D} \left[\zeta_R(D) - \frac{1}{2}f_D(z/L) \right] \quad (17)$$

while for the transverse components we find

$$\langle E_i^2 \rangle = -2 \frac{(D-2)\Gamma(D/2)}{(4\pi)^{D/2}L^D} \left[\zeta_R(D) + \frac{1}{2}f_D(z/L) \right]. \quad (18)$$

For the magnetic fluctuations we similarly have

$$\langle B_{iz}^2 \rangle = -(D-2) \frac{(D-2)\Gamma(D/2)}{(4\pi)^{D/2}L^D} \left[\zeta_R(D) - \frac{1}{2}f_D(z/L) \right] \quad (19)$$

and

$$\langle B_{i<j}^2 \rangle = (D-3) \frac{(D-2)\Gamma(D/2)}{(4\pi)^{D/2}L^D} \left[\zeta_R(D) + \frac{1}{2}f_D(z/L) \right]. \quad (20)$$

Note again that in these expressions we have summed over the transverse indices i and j , each taking $D-2$ different values. All these correlators are seen to diverge near the plates $z \rightarrow 0$ or $z \rightarrow L$ where the function $f_D(z/L)$ diverges. This is the same phenomenon which has previously been seen in $D=4$ dimensions [3].

The pressure between the plates due to these fluctuations is defined by $P = \langle T_{zz} \rangle$. From (1) we have $T_{zz} = B_{iz}^2 - E_z^2 + \mathcal{L}$ where now

$$\langle \mathcal{L} \rangle = -\frac{1}{2}(D-1) \frac{(D-2)\Gamma(D/2)}{(4\pi)^{D/2}L^D} f_D(z/L). \quad (21)$$

Together with the values for $\langle E_z^2 \rangle$ and $\langle B_{iz}^2 \rangle$ from above, the z -dependence from the function $f_D(z/L)$ cancels out in the pressure and gives the expected value (2).

So far there are no inconsistencies in the obtained results. But when we now calculate the energy density $\mathcal{E} = \langle T_{00} \rangle$ between the plates, with $T_{00} = E_i^2 + E_z^2 - \mathcal{L}$, we obtain

$$\mathcal{E} = -\frac{(D-2)\Gamma(D/2)}{(4\pi)^{D/2}L^D} [\zeta_R(D) - (D/2 - 2)f_D(z/L)]. \quad (22)$$

The z -dependence in the last term is non-zero when $D > 4$ and makes the energy density diverge like z^{-D} with distance z from the plates. As a result, the total energy of the system is infinite, a result which seems to be impossible to reconcile with the finite Casimir force (2). In fact, (2) corresponds to having a constant energy density equal to the first term in (22). This apparent inconsistency has been verified in a different approach based on Green's function methods [8].

It is tempting to explain this problem by the imposed boundary conditions. We have used the MIT boundary condition which is equivalent to letting the electromagnetic vector potential satisfy Neumann boundary conditions in the axial gauge. Had we instead imposed metallic boundary conditions, equivalent to Dirichlet boundary conditions for the vector potential in the axial gauge, the only change in the above results would be the replacement of the mode functions $\cos(n\pi z/L)$ with $\sin(n\pi z/L)$ in (4) so that $f_D \rightarrow -f_D$ in the above results. Needless to say, the problem would remain. Only for periodic boundary conditions, as for finite temperature, would the disturbing term be absent [9]. But this is not necessarily satisfying from a physical point of view. A more mathematical discussion of such divergences near confining boundaries has been initiated by Fulling but here only scalar fields are considered [10].

A physical explanation of the above conundrum becomes apparent when we take the limit $L \rightarrow \infty$ and thus consider the quantum fluctuations around a single plate. From (22) we then find the energy density

$$\mathcal{E}_1 = (D - 2)(D/2 - 2) \frac{\Gamma(D/2)}{(4\pi)^{D/2} |z|^D} \tag{23}$$

which is non-zero on both sides of the plate and diverges when we approach it. This situation is analogous to the diverging energy density surrounding a pointlike electron. It is intrinsic to a single plate and should not contribute to the interaction between the plates induced by the same vacuum fluctuations. To see the connection with the Casimir force, we should subtract the self-energy (23) for both plates from the full energy density (22), taking into account both sides of each plate. We thus obtain the interaction energy density

$$\tilde{\mathcal{E}} = -(D - 2) \frac{\Gamma(D/2)}{(4\pi)^{D/2} L^D} \times \begin{cases} (D/2 - 2)(L/(L - z))^D & \text{for } z < 0, \\ \zeta_R(D) - (D/2 - 2)\tilde{f}_D(z/L) & \text{for } 0 < z < L, \\ (D/2 - 2)(L/z)^D & \text{for } z > L, \end{cases} \tag{24}$$

where now

$$\tilde{f}_D(z/L) = \zeta_H(D, 1 + z/L) + \zeta_H(D, 2 - z/L). \tag{25}$$

It is seen to be finite everywhere, even at the plates. When integrating over the full volume, the z -dependent terms cancel out as follows from

$$\begin{aligned} & \int_{-\infty}^0 \frac{dx}{(1-x)^D} - \int_0^1 dx \tilde{f}(x) + \int_1^\infty \frac{dx}{x^D} \\ &= \frac{2}{D-1} \left[1 + \sum_{n=0}^\infty \left(\frac{1}{(n+2)^{D-1}} - \frac{1}{(n+1)^{D-1}} \right) \right] = 0. \end{aligned} \tag{26}$$

Only the z -independent term in (24) contributes and agrees perfectly with the total energy corresponding to the Casimir force.

A similar and somewhat simpler system is the Casimir energy induced by a massless scalar field in the same geometry. One will then find a very similar result for the energy density as obtained here [8]. It diverges near the plates for all spacetime dimensions $D > 2$. Again this can be attributed to a divergent self-energy of each plate. However, when $D = 2$ there are no such divergences and zero self-energy. But this is also the dimension in which the scalar theory has conformal invariance. In higher dimensions $D > 2$ it is possible to make the scalar theory retain this invariance by adding a conformal term. The resulting, improved energy-momentum tensor [12] then contains an additional piece discovered by Huggins [13] and makes it traceless. Including the Huggins term, the divergent part of the energy

density corresponding to the last term in (22) drops out as was first noticed by de Witt when $D = 4$ [11].

For the electromagnetic field we have used the canonical energy–momentum tensor (1) which has the trace $T_{\mu}^{\mu} = (4 - D)\mathcal{L}$. It is zero for $D = 4$ which reflects the well-known fact that the Maxwell theory is then conformally invariant. There are then no divergences in the Casimir energy. Thus it is natural to relate the apparent inconsistency in the electromagnetic Casimir energy when $D > 4$ to the lack of conformal invariance. It does not seem to be possible to construct an improved energy–momentum tensor in this case because gauge invariance forbids the existence of any corresponding local Huggins term. From this point of view the divergent, electromagnetic self-energy can therefore not be removed. For this to be done, one needs a more realistic description of the boundary plates along the lines considered by others [4].

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